Definition: Let $\langle M, \rho \rangle$ be a metric space and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in M. We say that s_n approaches to L (where $L \in M$) as n approaches infinity if given $\varepsilon > 0$ there exists $N \in I$ such that

$$\rho(s_n, L) < \varepsilon \quad (n \geq N).$$

In this case we write $\lim\limits_{n \to \infty} s_n = L$ (or)

$$s_n \to L \text{ as } n \to \infty,$$ and say that $\{s_n\}_{n=1}^{\infty}$ is convergent in M to the point L.

Defn Let $\langle M, \rho \rangle$ be a metric space and let $\{s_n\}_{n=1}^{\infty}$ be a sequence of points in M. We say that $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence if given $\varepsilon > 0$ there exists $N \in I$ such that

$$\rho(s_m, s_n) < \varepsilon \quad (m, n \geq N).$$

Theorem: Let $\langle M, \rho \rangle$ be a metric space. If $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of points of M, then $\{s_n\}_{n=1}^{\infty}$ is Cauchy.

Proof: Given $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence. (i.e.) $\lim\limits_{n \to \infty} s_n = L$

By defn given $\varepsilon > 0$ there exists $N_1 \in I$ such that

$$\rho(s_n, L) < \varepsilon/2 \quad \forall\, n \geq N_1 \quad \text{——— (1)}$$

also $\lim\limits_{m \to \infty} s_m = L$, given $\varepsilon > 0$ $\exists\, N_2 \in I$ such that

$$\therefore \rho(s_m, L) < \varepsilon/2 \quad \forall\, m \geq N_2 \quad \text{——— (2)}$$

let $N = \max\{N_1, N_2\}$

$\therefore \forall\, m, n \geq N$

$\rho(s_n, L) < \varepsilon/2$ and $\rho(s_m, L) < \varepsilon/2$ — (3)

$\forall\, m, n \geq N$

consider

$$\rho(s_m, s_n) \leq \rho(s_m, L) + \rho(L, s_n)$$

$$< \varepsilon/2 + \varepsilon/2 \quad \forall\, m, n \geq N$$

$$\rho(s_m, s_n) < \varepsilon \quad \forall\, m, n \geq N$$

$\Rightarrow$ seq $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

UNIT – V

5. Continuous functions on metric space.

5.1 Functions Continuous at a point on the real line.

5.1.A Definition: We say that the function f is continuous at $a \in R^1$ if $\lim_{x \to a} f(x) = f(a)$.

5.1B Theorem: If the real-valued functions f and g are continuous at $a \in R^1$ then
(i) $f+g$ (ii) $f-g$ (iii) fg (iv) f/g if $g(a) \neq 0$ are continuous at a.

Proof: given f and g are continuous at $a \in R^1$

$\Rightarrow \lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$

$\lim_{x\to a} f(x) = f(a)$ since f is a real valued function and $a \in R^1$

given $\epsilon > 0$ $\exists$ $\delta_1 > 0$ such that

$|f(x) - f(a)| < \epsilon/2$ $\quad 0 < |x-a| < \delta_1$. —— (1)

$\lim_{x\to a} g(x) = g(a)$

given $\epsilon > 0$ $\exists$ $\delta_2 > 0$ such that

$|g(x) - g(a)| < \frac{\epsilon}{2}$ $\quad 0 < |x-a| < \delta_2$ —— (2)

Let $\delta = \min\{\delta_1, \delta_2\}$.

when $0 < |x-a| < \delta$ then $\left.\begin{array}{l} |f(x) - f(a)| < \frac{\epsilon}{2} \\ \text{and } |g(x) - g(a)| < \frac{\epsilon}{2} \end{array}\right\}$ —— (3)

$\therefore$ when $0 < |x-a| < \delta$

Consider

$$|(f(x)+g(x)) - (f(a)+g(a))| = |(f(x) - f(a)) + (g(x) - g(a))|$$

$$\leq |f(x) - f(a)| + |g(x) - g(a)|$$

$$< \epsilon/2 + \epsilon/2$$

$$|(f(x)+g(x)) - (f(a)+g(a))| < \epsilon$$

$|(f+g)(x) - (f+g)(a)| < \epsilon$ when $0 < |x-a| < \delta$.

$\Rightarrow \lim_{x\to a} (f+g)(x) = (f+g)(a)$

$\Rightarrow f+g$ is continuous at $a \in R^1$

when $0 < |x-a| < \delta$

consider

$$|(f(x)-g(x))-(f(a)-g(a))| = |(f(x)-f(a))+(g(a)-g(x))|$$

$$\leq |f(x)-f(a)| + |g(a)-g(x)|$$

$$\leq |f(x)-f(a)| + |g(x)-g(a)|$$

$$< \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} \quad \text{using (3)}$$

$$|(f-g)(x)-(f-g)(a)| < \varepsilon_1 \quad \text{when } 0<|x-a|<\delta$$

$$\Rightarrow \lim_{x \to a} (f-g)(x) = (f-g)(a)$$

$\Rightarrow$ $f-g$ is Continuous at $a \in R^1$.